Differences and similarities between fundamental and adjoint matters in $SU(N)$ gauge theories

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We investigate differences and similarities between fundamental fermions and adjoint fermions in $SU(N)$ gauge theories. The gauge theory with fundamental fermions possesses $\mathbb{Z}_N$ symmetry only in the limit of infinite fermion mass, whereas the gauge theory with adjoint fermions does have the symmetry for any fermion mass. The flavor-dependent twisted boundary condition (FTBC) is then imposed on fundamental fermions so that the theory with fundamental fermions can possess $\mathbb{Z}_N$ symmetry for any fermion mass. We show similarities between FTBC fundamental fermions and adjoint fermions, using the Polyakov-loop extended Nambu–Jona-Lasinio (PNJL) model. In the mean-field level, the PNJL model with FTBC fundamental fermions has dynamics similar to the PNJL model with adjoint fermions for the confinement/deconfinement transition related to $\mathbb{Z}_N$ symmetry. The chiral property is somewhat different between the two models, but there is a simple relation between chiral condensates in the two models. As an interesting high-energy phenomenon, a possibility of the gauge symmetry breaking is studied for FTBC fundamental fermions.

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I. INTRODUCTION

Understanding of nonperturbative nature of quantum chromodynamics (QCD) is one of the most important subjects in particle physics. QCD has $\mathbb{Z}_N$ symmetry only in the limit of infinite current quark mass (m) and chiral symmetry only in the limit of $m = 0$. In the real world where m is finite, some nonperturbative properties are discovered by lattice QCD (LQCD). For finite temperature (T), for example, deconfinement and chiral transitions are found to be crossover [1]. For finite quark chemical potential (μ_q), however, our understanding of nonperturbative nature of QCD is still far from perfection, since LQCD simulations have the sign problem.

QCD is a $SU(3)$ gauge theory with fundamental fermions. In this sense, a $SU(N)$ gauge theory with adjoint fermions [2] is a QCD-like theory. This QCD-like theory is quite interesting, since it has $\mathbb{Z}_N$ symmetry for any m and no sign problem for finite μ_q when the number of flavor N_F is even. Furthermore, if the gauge theory with adjoint fermions is considered on spacetime $R^4 \times S^1$, there is a possibility that the theory has the Hosotani mechanism [3] where the gauge symmetry breaking (GB) is induced by the non-zero vacuum expectation value of the gauge component in a compact dimension S^1. Actually, the GB is found to occur, when the periodic boundary condition is imposed on adjoint fermions; see Ref. [4] and references therein. In the case of four-dimensional gauge theory at finite T where the spacetime of the theory is compactified into $R^3 \times S^1$, exotic phases such as the reconfined phase appear at high T , when adjoint fermions are introduced with the periodic boundary condition [5–7]. These interesting properties of adjoint matter lead us to an important question, how close is adjoint matter to fundamental matter?

The fermion number is $N \times N_F$ for N_F-flavor fundamental fermions and $N^2 - 1$ for one-flavor adjoint fermions. These numbers are almost identical with each other for $N = N_F \gg 1$, and even in the realistic case of $N = N_F = 3$ they are still close to each other. A difference is that the $SU(N)$ gauge theory with fundamental fermions does not possess $\mathbb{Z}_N$ symmetry for finite m. This difference can be removed by imposing the flavor-dependent twist boundary condition (FTBC) on fundamental fermions [8, 9]. We refer to fundamental fermions with the FTBC as FTBC fundamental fermions in this paper. For zero T, FTBC fundamental fermions yield the same dynamics as ordinary fundamental fermions with the anti-periodic boundary condition, since the fermion boundary condition does not affect dynamics there. In Refs. [8, 9], properties of the $SU(N)$ gauge theory with FTBC fundamental fermions were investigated with the Polyakov-loop extended Nambu-Jona-Lasinio (PNJL) model [7, 10–24]. The PNJL model with FTBC fundamental fermions has $\mathbb{Z}_N$ symmetry for any m. In the model, the symmetry is preserved at low T but spontaneously broken at high T, and the restoration of chiral symmetry is rather slow. Similar properties are seen in the $SU(3)$ gauge theory with adjoint fermions. These results imply that the $SU(N)$ gauge theory with FTBC fundamental fermions has dynamics similar to the $SU(N)$ gauge theory with adjoint fermions and hence the GB takes place also in the former theory.

In this paper, we show similarities between adjoint matter and FTBC fundamental matter in $SU(N)$ gauge theories, particularly for the confinement/deconfinement transition related to $\mathbb{Z}_N$ symmetry, using the PNJL model. This leads to the

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important conclusion that an essential difference between ordinary fundamental matter and adjoint matter is originated in the presence or absence of $\mathbb{Z}_N$ symmetry. Meanwhile, the chiral property is somewhat different between adjoint matter and FTBC fundamental matter, but we show that there is a simple relation between chiral condensates in the two matters. As an interesting high-energy phenomenon, a possibility of the GB is also examined for FTBC fundamental fermion and the result is compared with that for adjoint fermion.

This paper is organized as follows. In Sec. II, thermodynamics of a gauge theory with adjoint fermion and a gauge theory with FTBC fundamental fermion are constructed in a similar manner. In Sec. III, similarities between FTBC fundamental and adjoint matters are numerically analyzed with the PNJL model. In Sec. IV, a possibility of the GB is discussed for FTBC fundamental matter by using the one-loop effective potential and the result is compared with that for adjoint matter. Section V is devoted to a summary.

II. ADJOINT AND FTBC FUNDAMENTAL FERMIONS

In this section, thermodynamics of a $SU(N)$ gauge theory with adjoint fermions and a $SU(N)$ gauge theory with FTBC fundamental fermions [8, 9] are constructed in a similar way. For later convenience, we start with a $SU(N)$ gauge theory with fermions in the $N\times\bar{N}$ dimensional representation consisting of the N^2-1 dimensional adjoint and the one-dimensional singlet representation. Here we consider a general case of $N_{F,adj}$ degenerate flavors, where $N_{F,adj}$ means the number of flavors for adjoint fermions. In Euclidean spacetime, the Lagrangian density L_{adj} becomes

$$L_{adj} = N_{F,adj}\bar{\Psi}(\gamma_\nu D_\nu^{N\times\bar{N}} + m)\Psi + \frac{1}{4g^2}F_{\mu\nu}^{a\ 2} \tag{1}$$

with $D_\nu^{N\times\bar{N}} \equiv \partial_\nu - i(A_\nu + \tilde{A}_\nu) = \partial_\nu - iA_{a,\nu}(t_a - \tilde{t}_a)$. The fermion field Ψ in its color part is described as a direct product of ψ_c and $\tilde{\psi}_c$ ($c = 1, 2, \cdots, N$), where ψ_c ($\tilde{\psi}_c$) is transformed as the N ($\bar{N}$) dimensional representation and the generator t_a ($\tilde{t}_a$) acts only on ψ_c ($\tilde{\psi}_c$). Below we consider a general temporal boundary condition for Ψ:

$$\Psi(\tau = \beta, \mathbf{x}) = e^{i\varphi}\Psi(\tau = 0, \mathbf{x}) \tag{2}$$

with Euclidean time τ and $\beta = 1/T$. The angle φ parameterizes the boundary condition. A value of $\varphi = \pi$ (0) corresponds to the anti-periodic (periodic) boundary condition.

The Lagrangian density (1) is invariant under the large gauge transformation,

$$\begin{aligned}
\Psi &\to \Psi' = U\tilde{U}\Psi, \\
A_\nu &\to A'_\nu = UA_\nu U^{-1} + i(\partial_\nu U)U^{-1}, \\
\tilde{A}_\nu &\to \tilde{A}'_\nu = \tilde{U}\tilde{A}_\nu\tilde{U}^{-1} + i(\partial_\nu\tilde{U})\tilde{U}^{-1},
\end{aligned} \tag{3}$$

where

$$U(x,\tau) = \exp{(i\alpha_a t_a)}, \tag{4}$$
$$\tilde{U}(x,\tau) = \exp{(-i\alpha_a \tilde{t}_a)}, \tag{5}$$

are elements of the SU(N) group characterized by real functions $\alpha_a(x,\tau)$ and satisfy the boundary conditions

$$U(x,\beta) = \exp{(-i2\pi k/N)}U(x,0), \tag{6}$$
$$\tilde{U}(x,\beta) = \exp{(i2\pi k/N)}\tilde{U}(x,0) \tag{7}$$

with integer k. In this transformation, $\tilde{\psi}_c$ is transformed as the conjugate representation of ψ_c. Thus Ψ belongs to the $N\times\bar{N}$ dimensional representation that is decomposed into the N^2-1 dimensional adjoint and the one-dimensional singlet representation. This theory is finally reduced to a $SU(N)$ gauge theory with adjoint fermions, since the singlet fermion is decoupled from the adjoint ones. The gauge transformation (3) includes the $\mathbb{Z}_N$ transformation. The Lagrangian density (1) and the fermion boundary condition (2) are invariant under the $\mathbb{Z}_N$ transformation, since the $\mathbb{Z}_N$ transformation on ψ_c is canceled out by that on $\tilde{\psi}_c$. Hence $\mathbb{Z}_N$ symmetry is exact in this theory.

Now a gauge theory with FTBC fundamental fermions is constructed by replacing the color-dependent fields $\tilde{\psi}_c$ ($c = 1, 2, \cdots, N$) by the flavor-dependent ones $\hat{\psi}_f$ ($f = 1, 2, \cdots, N$) in Ψ and the gauge field $i\tilde{A}_{\nu,a}$ by the flavor-dependent imaginary chemical potential $i\hat{B}_\nu = i\Theta_a\hat{t}_a\delta_{\nu,4}T$ with the $SU(N)$ generators $\hat{t}_a$ ($a = 1, 2, \cdots, N^2-1$) and the unit matrix $\hat{t}_0$ in flavor space and real parameters Θ_a ($a = 0, 1, \cdots, N^2-1$). After the replacement, the new fermion field Ψ_{fund} belongs to the N dimensional fundamental representation in color space. Hence one can represent $\Psi_{fund} = (\Psi_1, \cdots, \Psi_N)^T$ with fundamental fermions Ψ_f labeled by flavors $f = 1, \cdots, N$. This replacement thus changes $N\times\bar{N}$ dimensional fermions of $N_{F,adj}$ flavors to fundamental fermions Ψ_f of $N_{F,fund} = N_{F,adj}N$ flavors. The Lagrangian density L_{fund} is then

$$L_{fund} = N_{F,adj}\bar{\Psi}_{fund}(\gamma_\nu D_\nu^N + m)\Psi_{fund} + \frac{1}{4g^2}F_{\mu\nu}^{a\ 2}, \tag{8}$$

where $D_\nu^N \equiv \partial_\nu - i(A_\nu + \hat{B}_\nu)$. This is nothing but a $SU(N)$ gauge theory with $N_{F,fund}$ flavor fundamental fermions with the flavor-dependent imaginary chemical potential. Although the gauge field A_ν is decoupled from the flavor degrees of freedom, one can consider the following baryon-number-color-flavor linked (BCFL) transformation

$$\begin{aligned}
\Psi_{fund} &\to \Psi'_{fund} = U\hat{U}\Psi_{fund} \\
A_\nu &\to A'_\nu = UA_\nu U^{-1} + i(\partial_\nu U)U^{-1}
\end{aligned} \tag{9}$$

where

$$U(x,\tau) = \exp{(i\alpha_a t_a)} \tag{10}$$
$$\hat{U}(\tau) = \exp{(i2\pi kT\tau/N)}(\hat{U}_{\rm Sf})^k, \tag{11}$$

where U is the same as in (4), but $\hat{U}$ with integer k acts on flavor space through a factor $(\hat{U}_{\rm Sf})_{ff'} = \delta_{f-1,f'}$ for $f\neq 1$ and $(\hat{U}_{\rm Sf})_{ff'} = \delta_{N,f'}$ for $f = 1$; note that $U_{\rm Sf}$ is the $N\times N$ unitary matrix by which flavor labels are shifted from f to $f-1$ for $f\neq 1$ and from f to N for $f = 1$. The matrix

$U_{\rm Sf}$ in flavor space is equal to $i^s U_{\rm SUf}$ with $s = 1$ for even N and $s = 0$ for odd N and an element $U_{\rm SUf}$ of the flavor $SU(N)$ group. The transformation (9), characterized by the $\mathbb{Z}_N$ transformation parameter k, thus links baryon number, color and flavor.

The boundary condition for Ψ_{fund} at $\tau = \beta$ is invariant under the transformation (9), but the Lagrangian density (8) is changed into

$$L_{fund} = N_{F,adj}\bar{\Psi}_{fund}(\gamma_\nu D_\nu^{N\prime} + m)\Psi_{fund} + \frac{1}{4g^2}{F_{\mu\nu}^{a}}^2, \tag{12}$$

where $D_\nu^{N\prime} = \partial_\nu - i(A_\nu + \hat{B}'_\nu)$ with $\hat{B}'_\nu = \Theta'_a \hat{t}_a \delta_{\nu,4} T$ and

$$\Theta'_a \hat{t}_a = (\hat{U}_{\rm Sf}^{-1})^k (\Theta_a \hat{t}_a - \frac{2k\pi}{N}\hat{t}_0)(\hat{U}_{\rm Sf})^k. \tag{13}$$

In general, L_{fund} is not invariant under this transformation. However, when

$$\begin{aligned} \Theta_a \hat{t}_a &= {\rm diag}(\theta_1, \theta_2, \cdots, \theta_N) \\ &= {\rm diag}(0, 2\pi/N, \cdots, \\ &\quad 2(l-1)/\pi/N, \cdots, 2\pi(N-1)) \end{aligned} \tag{14}$$

with $l = 1, 2, \cdots, N$, the transformed imaginary chemical potential $\Theta'_a \hat{t}_a$ becomes

$$\begin{aligned} \Theta'_a \hat{t}_a &= (\theta'_1, \theta'_2, \cdots, \theta'_N) \\ &= (U_{\rm Sf}^{-1})^k {\rm diag}(-2\pi k/N, 2\pi(1-k)/N, \cdots, \\ &\quad 2\pi(l-k-1)/N, \cdots, 2\pi(N-k-1))(U_{\rm Sf})^k \\ &\cong {\rm diag}(0, 2\pi/N, \cdots, \\ &\quad 2(l-1)/\pi/N, \cdots, 2\pi(N-1)) \\ &= (\theta_1, \theta_2, \cdots, \theta_N) = \Theta_a \hat{t}_a, \end{aligned} \tag{15}$$

where $\theta'_f \cong \theta_f$ means $\theta'_f/(2\pi) = \theta_f/(2\pi) \pmod 1$; note that θ_f have a trivial periodicity of 2π in the QCD partition function. $\mathbb{Z}_N$ symmetry is thus exact in a $SU(N)$ gauge theory with fundamental fermions of $N_{F,fund}$ degenerate flavors, when the flavor-dependent imaginary chemical potential of (14) is introduced. Obviously, the flavor-dependent imaginary chemical potential breaks flavor symmetry partially. In the confinement phase, flavor symmetry is recovered from the breaking [8, 9], as shown explicitly in Sec. III.

When the fermion fields Ψ_f are transformed as

$$\Psi_f \to \exp{(i\theta_f T\tau)}\Psi_f \tag{16}$$

for $f = 1, 2, \cdots, N$, the Lagrangian (8) with the imaginary chemical potential (14) is changed into a new Lagrangian density with no imaginary chemical potential,

$$\begin{aligned} L_{fund} &= N_{adj}\bar{\Psi}_{fund}(\gamma_\nu(\partial_\mu - iA_\nu) + m)\Psi_{fund} \\ &\quad + \frac{1}{4g^2}{F_{\mu\nu}^{a}}^2, \end{aligned} \tag{17}$$

with the boundary condition

$$\Psi_f(\tau = \beta, \mathbf{x}) = e^{i(\varphi - \theta_f)}\Psi_f(\tau = 0, \mathbf{x}). \tag{18}$$

The boundary condition (18) with the twist angles θ_f of (14) is the flavor-dependent boundary condition (FTBC) and fundamental fermions satisfying the FTBC are FTBC fundamental fermions. As mentioned above, the $SU(N)$ gauge theory with FTBC fundamental fermions has $\mathbb{Z}_N$ symmetry just as the $SU(N)$ gauge theory with adjoint fermions. The fermion number, i.e., the product of the color and flavor numbers, is $N_{F,adj}N^2$ for the former theory and $N_{F,adj}(N^2-1)$ for the latter one. These numbers are close to each other even for $N = 3$. Thus there is a possibility that the two theories have similar dynamics to each other. This will be tested in the following two sections; low energy dynamics is analyzed in Sec. III, while the GB as a nontrivial high-energy phenomenon is investigated in Sec. IV.

The adjoint representation is a real representation under the gauge transformation and hence the system has no sign problem even at real μ_q, when $N_{F,adj}$ is even. The fundamental representation with the FTBC, meanwhile, is not real and therefore the system has a sign problem at real μ_q. However, if we restrict the color gauge transformation U on its center group $\mathbb{Z}_N$ in the color-flavor linked transformation (9), the fermion field Ψ_{fund} becomes real under the restricted transformation at $\tau = \beta$. In fact, Ψ_{fund} and its charge conjugation Ψ_{fund}^C are transformed under the restricted transformation as $\Psi_{fund} \to (U_{\rm Sf})^k \Psi_{fund}$ and $\Psi_{fund}^C \to (U_{\rm Sf})^k \Psi_{fund}^C$ at $\tau = \beta$. Obviously, this reality of Ψ_{fund} at $\tau = \beta$ ensures $\mathbb{Z}_N$ symmetry to be exact.

Now we consider the chiral limit of $m = 0$ as an ideal case and take the case of $N = N_{F,fund} = 3$ for a typical example. The present chiral- and $\mathbb{Z}_3$-symmetric $SU(3)$ gauge theory with FTBC fundamental fermions was constructed from the $SU(3)$ gauge theory in which fundamental fermions have $SU(3)_{\rm R}$ and $SU(3)_{\rm L}$ flavor symmetries. The $SU(3)_{\rm R}$ and $SU(3)_{\rm L}$ symmetries are broken down to $(U(1)_{\rm R})^2$ and $(U(1)_{\rm L})^2$ symmetries, respectively, by introducing the FTBC, i.e., the flavor-dependent imaginary chemical potential. These symmetries remain as continuous symmetries in addition to global $U(1)_{\rm B}$ symmetry. Thus, instead of the preservation of $\mathbb{Z}_3$ symmetry, the chiral symmetry is partially broken. This implies that there is a NO-GO theorem on construction of the chiral- and $\mathbb{Z}_N$-symmetric $SU(N)$ gauge theory with ordinary fundamental fermions. It seems that it is impossible to construct a fully chiral symmetric and $\mathbb{Z}_N$ symmetric $SU(N)$ gauge theory with fundamental fermions without introducing the additional gauge field that plays the role of the conjugate gauge field $\tilde{A}_\nu$ in the $SU(N)$ gauge theory with adjoint fermions and is coupled to other charges such as flavor or baryon number. Several properties of gauge theories with ordinary fundamental, FTBC fundamental and adjoint fermions are summarized in Table I.

Instead of the flavor-dependent imaginary chemical potential, one may consider the color-dependent imaginary chemical potential $i\theta^C T$ with

$$\begin{aligned} \theta^C &= {\rm diag}(\theta_1^C, \theta_2^C, \cdots, \theta_N^C) \\ &= {\rm diag}(0, 2\pi/N, \cdots, \\ &\quad 2(l-1)/\pi/N, \cdots, 2\pi(N-1)). \end{aligned} \tag{19}$$

property	Fundmental	FTBC	Adjoint
$\mathbb{Z}_N$	broken	symmetric	symmetric
Reality	not real	not real	real
Flavor	symmetric	partially broken	symmetric
Chiral	symmetric	partially broken	symmetric
Sign problem	exist	exist	none for even $N_{F,adj}$

TABLE I: Summary of properties of $SU(N)$ gauge theories with ordinary fundamental, FTBC fundamental and adjoint fermions. Chiral symmetry is considered in the case of $m = 0$.

In this case, the gauge symmetry is partially broken; for example, $SU(3)$ gauge symmetry is broken to $U(1)^2$ the generators of which are t_3 and t_8 in Cartan sub-algebra of $SU(3)$ group. However, the system is $\mathbb{Z}_3$ symmetric under the baryon-number-color linked (BCL) transformation

$$\begin{aligned}\Psi_{fund} &\rightarrow \Psi'_{fund} = UV\Psi_{fund} \\ A_\nu &\rightarrow A'_\nu = UA_\nu U^{-1} + i(\partial_\nu U)U^{-1}, \end{aligned} \tag{20}$$

where

$$V(\tau) = \exp\left(i2\pi kT\tau/3\right)(U_{\rm Sc})^k \tag{21}$$

and

$$U(x,\tau) = \exp[\{i(\alpha_3 t_3 + \alpha_8 t_8)] \tag{22}$$

with the temporary boundary condition

$$U(x,\beta) \;=\; \exp\left(-i2\pi k/3\right)U(x,0). \tag{23}$$

$$\tag{24}$$

The matrix $U_{\rm Sc}$ has the same form as $\hat{U}_{\rm Sf}$ but acts on color space. When τ is neither 0 nor β, U is an element of color $SU(N)$ group, but V is not. Therefore, this invariance is not trivial. Thus $\mathbb{Z}_3$ symmetry may appear by abandoning full $SU(3)$ gauge invariance.

Even in the theory with no color-dependent chemical potential, gauge symmetry is partially broken, if the temporal gauge field A_4 has an expectation value. In this situation, the expectation value acts just as the color-dependent imaginary chemical potential. At low energy where gauge interaction is strong, the GB mentioned above is expected to be canceled out by confinement [25]. At high energy where the gauge interaction is weak, in contrast, the breaking may appear. This is nothing but the Hosotani mechanism [3]. This possibility is discussed in Sec. IV particularly for the case of FTBC fundamental fermions.

III. LOW-ENERGY DYNAMICS

In this section, we analyze low-energy dynamics of $SU(3)$ gauge theories with FTBC and adjoint (ADJ) fermions and investigate similarities between them, using the Polyakov-loop extended Nambu-Jona-Lasinio (PNJL) model [7, 10–24]. The PNJL Lagrangian density with N_F degenerate flavors is obtained by

$$\mathcal{L}_{\rm PNJL} \;=\; \sum_f^{N_F} \bar{\Psi}_f(\gamma_\nu D_\nu^{\rm PNJL} + m)\Psi_f + \mathcal{L}_{\rm NJL} + \mathcal{U}, \tag{25}$$

where $D_\nu^{\rm PNJL} = \partial_\nu - i\delta_{\nu,4}A_4$. We use the Polyakov gauge in which $A_i = 0$ for $i = 1, 2, 3$ and A_4 is considered as a background field.

In (25), $\mathcal{L}_{\rm NJL}$ stands for effective quark-quark interactions. For fundamental quarks with $N = N_{F,fund} = 3$, they have the standard form of

$$\begin{aligned}\mathcal{L}_{{\rm NJL},fund} \;&=\; -G_{\rm S}\sum_{a=0}^{8}[(\bar{\Psi}\lambda_a\Psi)^2 + (\bar{\Psi}i\gamma_5\lambda_a\Psi)^2] \\ &\quad + G_{\rm D}\left[\det_{ff'}\bar{\Psi}_f(1+\gamma_5)\Psi_{f'} + {\rm h.c.}\right], \end{aligned} \tag{26}$$

where λ_a is the Gell-Mann matrix in flavor space and $G_{\rm S}$ and G_D are coupling constants of the scalar-type four-quark interaction and the Kobayashi-Maskawa-t'Hooft determinant interaction, respectively [26, 27]. Table II(a) shows values of the coupling constants in addition to the current quark mass m and the three-dimensional momentum cutoff Λ. The coupling constants and the cutoff are determined to reproduce empirical values of η'- and π-meson masses and π-meson decay constant at vacuum when $m_u = m_d = 5.5$MeV and $m_s = 140.7$MeV [28]. In this paper, however, we take symmetric current quark masses, $m_f = 5.5$MeV for any flavor, to make the system flavor symmetric at $\theta_f = 0$.

(a)	m_f(MeV)	Λ(MeV)	$G_{\rm s}\Lambda^2$	$G_{\rm D}\Lambda^5$	
	5.5	602.3	1.835	12.36	
(b)	m(MeV)	Λ(MeV)	$G\Lambda^2$		
	5.5	651	3.607		
(c)	a_0	a_1	a_2	b_3	T_0(MeV)
	3.51	-2.47	15.2	-1.75	195

TABLE II: Summary of the parameter set in the PNJL model: (a) parameters of the NJL sector for fundamental fermions with $N = 3$ and $N_{F,fund} = 3$, (b) parameters of the NJL sector for adjoint fermions with $N = 3$ and $N_{F,adj} = 1$, and (c) parameters of the Polyakov-loop potential for the case of $N = 3$.

As the NJL sector for ADJ fermion, we take the form of

$$\mathcal{L}_{{\rm NJL},adj} = -G(\bar{Q}Q)^2 \tag{27}$$

proposed in Ref. [22]; see Table II(b) for the parameter set in the case of $N = 3$ and $N_{F,adj} = 1$.

In (25), the Polyakov-loop potential $\mathcal{U}$ is a function of A_4. Particularly in the cases of $N = 2$ and 3, it can be written as a function of the Polyakov-loop Φ and its conjugate Φ^* [29]. In the fundamental representation, they are defined by

$$\Phi = \frac{1}{N}{\rm tr}_c(L_{fund}), \quad \Phi^* = \frac{1}{N}{\rm tr}_c(\bar{L}_{fund}), \tag{28}$$

where L_{fund} is the fundamental representation of $L = \exp(i\phi) = \exp(iA_4/T)$. In the Polyakov gauge, they are written as

$$\Phi = \frac{1}{N}(e^{i\phi_1} + e^{i\phi_2} + \cdots + e^{i\phi_N}), \tag{29}$$

where the ϕ_i satisfy the condition $\phi_1 + \phi_2 + \cdots + \phi_N = 0$.

For $N = 3$, we take the Polyakov-loop potential $\mathcal{U}$ of Ref. [15]:

$$\mathcal{U} = T^4\Big[-\frac{a(T)}{2}\Phi^*\Phi + b(T)\ln(1-6\Phi\Phi^* + 4(\Phi^3+\Phi^{*3}) - 3(\Phi\Phi^*)^2)\Big], \tag{30}$$

$$a(T) = a_0 + a_1\Big(\frac{T_0}{T}\Big) + a_2\Big(\frac{T_0}{T}\Big)^2, \quad b(T) = b_3\Big(\frac{T_0}{T}\Big)^3. \tag{31}$$

Parameters of $\mathcal{U}$ are fitted to LQCD data at finite T in the pure gauge limit. The Polyakov-loop potential yields the first-order deconfinement phase transition at $T = T_0$ in the pure gauge theory [30, 31]. The original value of T_0 is 270 MeV determined from the pure gauge LQCD data, but the PNJL model with this value yields a larger value of the pseudocritical temperature $T_{\rm c}$ at zero chemical potential than $T_c \approx 160$ MeV predicted by full LQCD [32–34]. We then rescale T_0 to 195 MeV so as to reproduce $T_c \sim 160$ MeV [23]. Parameters in the Polyakov-loop potential are summarized in Table II(c), together with the other parameters.

For the case of $N_{F,fund} = N_N N$ $(N_N = 1, 2, \cdots)$ fundamental quarks with the FTBC, the thermodynamic potential $\Omega_{\rm FTBC}$ is obtainable from $\mathcal{L}_{\rm PNJL}$ with the mean-field approximation:

$$\Omega_{\rm FTBC} = N_N \Omega_{q,{\rm FTBC}} + U_{fund} + \mathcal{U} \tag{32}$$

with

$$\Omega_{q,{\rm FTBC}} = -2\sum_{c=1}^{N}\sum_{f=1}^{N}\int\frac{d^3p}{(2\pi)^3}\Big[E_f + \frac{1}{\beta}\ln\,[1+e^{i\phi_c}e^{i\theta_f}e^{-\beta E_{f-}}] + \frac{1}{\beta}\ln\,[1+e^{-i\phi_c}e^{-i\theta_f}e^{-\beta E_{f+}}]\Big] + U_{fund} + \mathcal{U}, \tag{33}$$

where $E_f = \sqrt{\mathbf{p}^2 + M_f^2}$ and $E_{f\pm} = \sqrt{\mathbf{p}^2+M_f^2} \pm i(\pi-\varphi)T$ for the constituent quark mass M_f. For $N = N_{F,fund} = 3$, the constituent quark mass M_f and the mesonic potential U_{fund} are given by

$$M_f = m_f - 4G_{\rm S}\sigma_f + 2G_{\rm D}\sigma_{f'}\sigma_{f''} \tag{34}$$

for $f \neq f'$ and $f \neq f''$ and $f' = f''$ and

$$U_{fund}^{N=N_{F,fund}=3} = \sum_{f=u,d,s} 2G_{\rm S}\sigma_f^2 - 4G_{\rm D}\sigma_u\sigma_d\sigma_s, \tag{35}$$

where σ_f $(f = u, d, s)$ is the expectation value of chiral condensate $\bar{\Psi}_f\Psi_f$.

For $N = 3$, when the summation is taken over color indices c, Eq. (33) is rewritten into

$$\Omega_{q,{\rm FTBC}}^{N=3} = -2\sum_{f=1}^{3}\int\frac{d^3p}{(2\pi)^3}\Big[3E_f + \frac{1}{\beta}\ln\,[1+3\Phi e^{i\theta_f}e^{-\beta E_{f-}} + 3\Phi^* e^{2i\theta_f}e^{-2\beta E_{f-}} + e^{3i\theta_f}e^{-3\beta E_{f-}}] + \frac{1}{\beta}\ln\,[1+3\Phi^* e^{-i\theta_f}e^{-\beta E_{f+}} + 3\Phi e^{-2i\theta_f}e^{-2\beta E_{f+}} + e^{-3i\theta_f}e^{-3\beta E_{f+}}].\Big] \tag{36}$$

If $\Phi = \Phi^* = 0$, this equation is further reduced to

$$\Omega_{q,{\rm FTBC}}^{N=3} = -2\sum_{f=1}^{3}\int\frac{d^3p}{(2\pi)^3}\Big[3E_f + \frac{1}{\beta}\ln\,[1+e^{-3\beta E_{f-}}] + \frac{1}{\beta}\ln\,[1+e^{-3\beta E_{f+}}]\Big]. \tag{37}$$

Equation (37) shows that red, blue and green quarks are statistically in the same state. The flavor-dependent imaginary chemical potential disappears in (37) for the confined phase of $\Phi = \Phi^* = 0$. The flavor-symmetry breaking due to the flavor-dependent imaginary chemical potential is thus dynamically restored in the confined phase. As a consequence of this property, Eq. (37) has a simpler form of

$$\Omega_{q,{\rm FTBC}}^{N=3} = -2\int\frac{d^3p}{(2\pi)^3}\Big[9E_{\rm con} + \frac{3}{\beta}\ln\,[1+e^{-i3(\pi-\varphi)}e^{-3\beta E_{\rm con}}] + \frac{3}{\beta}\ln\,[1+e^{i3(\pi-\varphi)}e^{-3\beta E_{\rm con}}]\Big], \tag{38}$$

with $M_{\rm con} \equiv M_u = M_d = M_s$ and $E \equiv \sqrt{\mathbf{p}^2 + M_{\rm con}^2}$ in confined phase.

Now we consider ADJ fermions with $N_{F,adj} = N_{F,fund}/N = N_N$ flavors. For the fermion boundary condition with φ, the thermodynamic potential is obtained with the mean field approximation as

$$\Omega_{adj} = N_N\Omega_{q,adj} + U_{adj} + \mathcal{U} \tag{39}$$

with

$$\Omega_{q,adj} = -2\sum_{c=1}^{N}\sum_{c'=1}^{N}\int \frac{d^3p}{(2\pi)^3}\Big[E + \frac{1}{\beta}\ln\ [1+e^{i\phi_c}e^{-i\phi_{c'}}e^{-\beta E_-}] + \frac{1}{\beta}\ln\ [1+e^{-i\phi_c}e^{i\phi_{c'}}e^{-\beta E_+}]\Big] - \Omega_1, \tag{40}$$

$$\Omega_1 = -2\int \frac{d^3p}{(2\pi)^3}\Big[E + \sum_{j=\pm}\frac{1}{\beta}\ln\ [1+e^{-\beta E_j}]\Big]\Big], \tag{41}$$

$M = m - 2G_{\rm S}\sigma$, $E = \sqrt{\mathbf{p}^2+M^2}$, $E_\pm = E \pm i(\pi-\varphi)T$ and $U_{adj} = G\sigma^2$ for the expectation value σ of chiral condensate $\bar{\Psi}\Psi$.

For $N = 3$, when the summation is taken over color indices c and c', Eq. (40) becomes [24]

$$\begin{aligned}\Omega^{N=3}_{q,adj} = -2\int \frac{d^3p}{(2\pi)^3}\Big[8E &+ \sum_{j=\pm}\frac{1}{\beta}\ln\ [1+(9\Phi\Phi^*-1)e^{-\beta E_j} \\ &+ (27\Phi^3+27\Phi^{*3}-27\Phi\Phi^*+1)e^{-2\beta E_j} \\ &+ (81\Phi^2\Phi^{*2}-27\Phi\Phi^*+2)e^{-3\beta E_j} \\ &+ (162\Phi^2\Phi^{*2}-54\Phi^3-54\Phi^{*3}+18\Phi\Phi^*-2)e^{-4\beta E_j} \\ &+ (81\Phi^2\Phi^{*2}-27\Phi\Phi^*+2)e^{-5\beta E_j} \\ &+ (27\Phi^3+27\Phi^{*3}-27\Phi\Phi^*+1)e^{-6\beta E_j} \\ &+ (9\Phi\Phi^*-1)e^{-7\beta E_j}+e^{-8\beta E_j}]\Big].\end{aligned} \tag{42}$$

The one-quark state, i.e., the term proportional to $e^{-\beta E_j}$ does not vanish in (42), even if $\Phi = \Phi^* = 0$. If $\Phi = \Phi^* = \pm 1/3$, meanwhile, the one-quark state vanishes and the Polyakov-loop Φ_{adj} in the adjoint representation, given by [24]

$$\Phi_{adj} = \frac{1}{8}{\rm tr}(L_{adj}) = \frac{1}{8}(9\Phi\Phi^*-1), \tag{43}$$

also vanishes, where L_{adj} are the adjoint representation of L. As seen below, however, this situation is not realized. Besides, even if it is realized, one can not consider a confinement with Φ_{adj}, since Φ_{adj} is invariant under $\mathbb{Z}_3$ transformation and hence it is not an order parameter of $\mathbb{Z}_3$ symmetry. We then consider a confinement as that of the static fundamental charge. For this definition of confinement, we can use Φ as an order parameter of a confinement/deconfinement transition. In fact, LQCD simulations [2] of a $SU(3)$ gauge theory with ADJ fermions indicate that below the critical temperature of the deconfinement transition a potential between static fundamental charges linearly raises with respect to increasing a distance R between the two charges, but a potential between static adjoint charges has a linearly-raising form only at small R. Thus a string breaking occurs at large R when static adjoint charges are taken.

In numerical calculations of the PNJL model, we take $N = 3$ and $N_3 = 1$, i.e., $N_{F,fund} = 3$ and $N_{F,adj} = 1$. The expectation values σ (or σ_f) and the ϕ_c are determined by the minimum condition of Ω. Figure 1(a) shows T dependence of Φ for FTBC fermion with $N = N_{F,fund} = 3$ and ADJ fermion with $N = 3$ and $N_{F,adj} = 1$. Here we take the anti-periodic boundary condition by setting $\varphi = \pi$. The two cases show similar T dependence: more precisely, Φ has a jump at $T = T_c \sim 195$MeV from 0 to 0.5. Below T_c, $\mathbb{Z}_3$ symmetry is surely preserved in both the cases. For the both cases, Φ never has $\pm 1/3$, since Φ has a jump from 0 to 0.5.

Figure 1(b) shows T dependence of the ϕ_c $(c = 1, 2, 3)$ at low T for the two cases after an appropriate relabeling of color indexes c. It is found that $\phi_c = 2\pi k/3$ with $k = 0, \pm 1$ at low T. The results for ADJ and FTBC fermions agree with each other. Therefore,

$$\phi_c^{adj} = \phi_c^{fund} = -\theta_f \tag{44}$$

holds true after an appropriate relabeling of c. This means that the back ground gauge field in (33) is the same as in (40). The two models thus have similar dynamics to each other at low T, as far as the confinement/deconfinement transition is concerned.

On the other hand, the chiral property is somewhat different between the two models. Figure 1(c) shows T dependence of M and M_f for the two cases. The constituent quark mass M of ADJ fermion is much larger than M_f of FTBC fermion. For FTBC fermion, furthermore, three degenerate quarks split into two heavy ones and light one at high T, since flavor symmetry is broken by the $\mathbb{Z}_3$-symmetry breaking. It is found from discussions in Sec. II that the differences in the chiral sector are originated in the presence or absence of fluctuations of the conjugate gauge field $\tilde{A}_\nu$ around its means value.

As mentioned above, the constituent quark mass M of adjoint fermion is much heavier than M_f of FTBC fermion at low T. However, there is a approximate scaling low $M/M_f \sim N_c$. This can be understood as follows. From the stationary conditions for σ and σ_f at $T = 0$, we can obtain the following equation,

$$\sigma = -\frac{N_{adj}}{\pi^2}\int_0^{\Lambda_{adj}} dpp^2\frac{M}{E} \sim -\frac{N_{adj}}{3\pi^2}\Lambda^3_{adj}, \tag{45}$$

$$\sigma_f = -\frac{N}{\pi^2}\int_0^{\Lambda_{fund}} dpp^2\frac{M_f}{E_f} \sim -\frac{N}{3\pi^2}\Lambda^3_{fund}, \tag{46}$$

where N_{adj} is the dimension of the adjoint representation and is related to N as $N_{adj} = N^2 - 1$. Since $\Lambda_{adj} \sim \Lambda_{fund}$, we obtain $M/M_f \sim \sigma/\sigma_f \sim N_{adj}/N \sim N = 3$. Therefore, M is approximately three times heavier than M_f. For FTBC fermion, only the three-quark state can survive in the confinement phase, as shown in (38). Therefore, the quark part of the thermodynamic potential has a small contribution and do not affect much the confinement/deconfinement transition in the two cases. This is the reason why the confinement/deconfinement transitions are similar to each other between ADJ and FTBC matters.

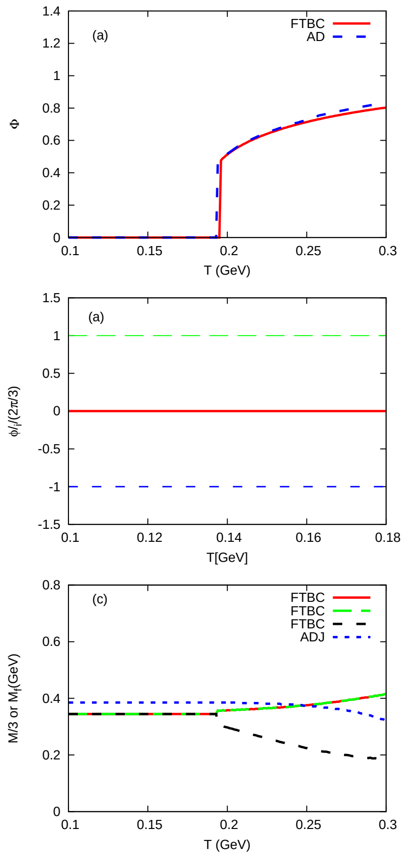

Fig. 1: T dependence of order parameters in the PNJL model of FTBC fermion with $N = N_{F,fund} = 3$ and that of ADJ fermion with $N = 3$ and $N_{F,adj} = 1$. In the both cases, the boundary condition $\varphi = \pi$ is taken. Three panels correspond to (a) Φ, (b) ϕ_c, and (c) $M/3$ and M_f, respectively.

Putting $\Phi = 0$ and neglecting higher-order terms of the suppression factor $e^{-\beta E}$, we obtain

$$\Omega^{N=3}_{q,adj} = -2 \int \frac{d^3p}{(2\pi)^3} \Big[8E + \frac{1}{\beta} [\ln\ [1 - e^{-i(\pi-\varphi)} e^{-\beta E}] + \frac{1}{\beta} \ln\ [1 - e^{i(\pi-\varphi)} e^{-\beta E}] \Big]. \tag{47}$$

Comparing (47) with (38) and noting $M \sim 3M_f$ in the confined phase, we see that the thermodynamic potential of ADJ fermion with $\varphi = 0$ (π) mimics that of FTBC fermion with $\varphi = \pi(0)$ in the confined phase, although coefficients of terms in (47) are somewhat different from those in (38). Since the fermion contribution itself is small in the confined phase, the difference between the periodic boundary condition (PB) and the anti-periodic boundary condition (APB) is negligibly small at low T, as shown in Fig. 2.

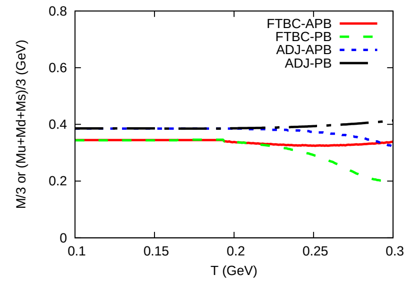

Fig. 2: T dependence of constituent quark masses in different types of fermions and boundary conditions. Four cases of FTBC-APB, FTBC-PB, ADJ-APB and ADJ-PB are taken, where FTBC and ADJ stand for kinds of fermions while PB and APB correspond to kinds of boundary conditions. Here we set $N = N_{F,fund} = 3$ and $N_{F,adj} = 1$. For FTBC-APB and FTBC-PB, constituent quark masses depend on flavor in the deconfinement phase at higher temperature, so the average vales are shown in the cases.

IV. GAGUE SYMMETRY BREAKING IN FTBC MODEL

As far as the confinement/deconfinement transition is concerned, there exist a similarity between FTBC and ADJ fermions even at high T, as shown in 1(a). This implies the possibility that the gauge symmetry breaking (GB) at higher temperature or higher-energy scale takes place also for FTBC fermion as a result of the Hosotani mechanism [3, 4]. In this section, we examine the GB in $SU(3)$ gauge theories with different types of fermions and boundary conditions on $R^3 \times S^1$ by using the one-loop effective potential. For simplicity, we regard the compact Euclidean time direction τ as a spacial direction y and replace $1/T$ by a size L of S^1. The gauge field is then obtained as

$$A_\mu = \langle A_y \rangle + A'_\mu, \tag{48}$$

where the VEV of A_y, $\langle A_y \rangle$, can be written as

$$\langle A_y \rangle = \frac{2\pi}{L} q \tag{49}$$

where $q = \mathrm{diag}(q_1, q_2, q_3)$ with $q_1 + q_2 + q_3 = 0$ and each component is determined as $(q_i)_{\mathrm{mod}\ 1}$. As in the previous section, the Polyakov-loop Φ is defined as

$$\Phi = \frac{1}{3}(e^{i2\pi q_1} + e^{i2\pi q_2} + e^{i2\pi q_3}) \tag{50}$$

Here we call the q_i "Polyakov-loop phases". When the solution is nontrivial, i.e., when q does not have a form at_0 for a constant a and the unit matrix t_0 in color space, the GB is expected to take place.

The one-loop effective potential $\mathcal{V}$ is a function of the q_i and consists of the gluon part $\mathcal{V}_g$ and the fermion part $\mathcal{V}_f$:

$$\mathcal{V} = \mathcal{V}_g + \mathcal{V}_f. \tag{51}$$

The gluon part $\mathcal{V}_g$ has an explicit form of

$$\mathcal{V}_g = -\frac{2}{L^4\pi^2} \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{n=1}^{\infty} \Big(1 - \frac{1}{3}\delta_{ij}\Big) \frac{\cos(2n\pi q_{ij})}{n^4} \tag{52}$$

with $q_{ij} = (q_i - q_j)_{\mathrm{mod}\ 1}$. For the case of fundamental fermion, $\mathcal{V}_f$ has a form of

$$\mathcal{V}_{f,fund} = \frac{2N_{F,fund}m^2}{L^2\pi^2}\sum_{i=1}^{3}\sum_{n=1}^{\infty}\frac{K_2(nmL)}{n^2} \times \cos[2\pi n(q_i+\varphi)] \quad (53)$$

for the fermion mass m and the boundary angle φ defined in (2), where K_2 is the modified Bessel function of the second kind. In the case of fundamental FTBC fermion, $\mathcal{V}_f$ is modified as

$$\mathcal{V}_{f,\mathrm{FTBC}} = \frac{2N_3m^2}{L^2\pi^2}\sum_{i=1}^{3}\sum_{f=1}^{3}\sum_{n=1}^{\infty} \times\frac{K_2(nmL)}{n^2}\cos[2\pi n(Q_{if}+\varphi)], \quad (54)$$

where $N_3 = N_{F,fund}/3$ and $Q_{if} = q_i + (f-1)/3$. In the case of ADJ fermion, meanwhile, $\mathcal{V}_f$ has a form of

$$\mathcal{V}_{f,adj} = \frac{2N_{F,adj}m^2}{L^2\pi^2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{n=1}^{\infty}\Big(1-\frac{1}{3}\delta_{ij}\Big) \times\frac{K_2(nmL)}{n^2}\cos[2\pi n(q_{ij}+\varphi)]. \quad (55)$$

Now we consider six combinations of different fermions and boundary conditions.

1. "FD-APB": fundamental fermions with anti-periodic boundary condition $\varphi = \pi$
2. "FD-PB": fundamental fermions with periodic boundary condition $\varphi = 0$
3. "ADJ-APB": adjoint fermions with anti-periodic boundary condition $\varphi = \pi$
4. "ADJ-PB": adjoint fermions with periodic boundary condition $\varphi = 0$
5. "FTBC-APB": FTBC fundamental fermions with the boundary condition $\varphi = \pi$
6. "FTBC-PB": FTBC fundamental fermions with the boundary condition $\varphi = 0$

First, we investigate the structure of $\mathcal{V}_f$ in the high-energy limit $mL \to 0$. Figure 3 shows the contour plot of $\mathcal{V}_f$ for the case of FD-APB. The fermion one-loop potential $\mathcal{V}_f$ becomes minimum at $(q_1, q_2, q_3) = (0, 0, 0)$ that gives $\Phi = 1$ and hence can not induce the GB. For the case of FD-PB shown in Fig.4, $\mathcal{V}_f$ has minima at $(q_1, q_2, q_3) = (\pm1/3, \pm1/3, \mp2/3)$ that lead to $|\Phi| = 1$; here $(\pm1/3, \pm1/3, \mp2/3)$ is a shorthand notation of $(1/3, 1/3, -2/3)$ and $(-1/3, -1/3, 2/3)$. Since $(\pm1/3, \pm1/3, \mp2/3) = (\pm1/3, \pm1/3, \pm1/3)_{\mathrm{mod}\ 1}$, the GB can not take place also in this case.

For the case of ADJ-APB, as shown in Fig. 5, $\mathcal{V}_f$ becomes minimum at $(q_1, q_2, q_3) = (0, 0, 0)$, $(\pm1/3, \pm1/3, \mp2/3)$ that gives $|\Phi| = 1$ and can not induce the GB. The fact that the

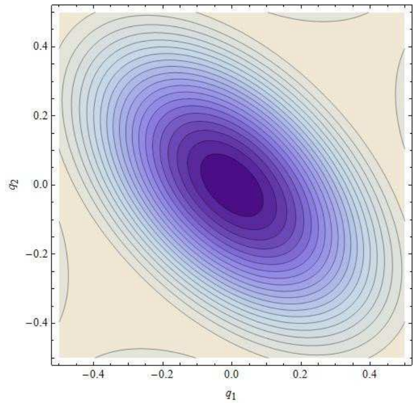

Fig. 3: Contour plot of $\mathcal{V}_f L^4$ in the limit $mL \to 0$ for the case of FD-APB. Here, q_3 is given by $-q_1 - q_2$.

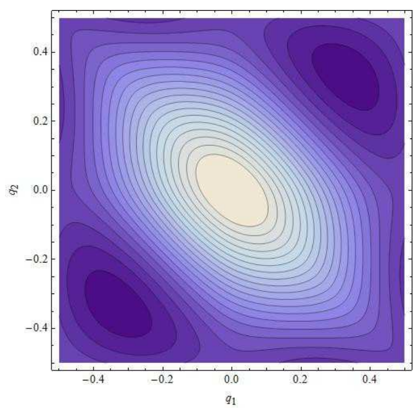

Fig. 4: The same figure as Fig. 3 but for the case of FD-PB.

three solutions are degenerate means that $\mathbb{Z}_3$ symmetry is preserved.

For the case of ADJ-PB, as shown in Fig. 6, $\mathcal{V}_f$ has minima at $(q_1, q_2, q_3) = (\pm1/3, \mp1/3, 0)$, $(\pm1/3, 0, \mp1/3)$ and $(0, \pm1/3, \mp1/3)$. These solutions are a family in the sense that $(\pm1/3, 0, \mp1/3)$ and $(0, \pm1/3, \mp1/3)$ are $\mathbb{Z}_3$ images of $(\pm1/3, \mp1/3, 0)$. Since the family yields $\Phi = 0$, this phase is sometimes called "re-confined phase". In the phase, the GB takes place and $SU(3)$ gauge symmetry is broken down to $U(1) \times U(1)$.

For the case of FTBC-APB, as shown in Fig. 7, $\mathcal{V}_f$ has minima at six points. The first $\mathbb{Z}_3$ family of $(q_1, q_2, q_3) = (0, 0, 0)$ and $(\pm1/3, \pm1/3, \mp2/3)$ leads to $|\Phi| = 1$, whereas the second $\mathbb{Z}_3$ family of $(q_1, q_2, q_3) = (\pm1/3, \mp1/3, 0)$, $(\pm1/3, 0, \mp1/3)$ and $(0, \pm1/3, \mp1/3)$ yields $\Phi = 0$. Thus $\mathbb{Z}_3$ symmetry is realized in FTBC fermions. The first family does not induce the GB, but the second family does.

One can see from Figs. 6 and 7 that near points yielding $\Phi = 0$ the structure of $\mathcal{V}_f$ is similar between ADJ-PB and FTBC-APB fermions. This result is consistent with that in the previous section. Near points yielding $\Phi = 1$, meanwhile, the structure of $\mathcal{V}_f$ is different between FTBC-APB and ADJ-

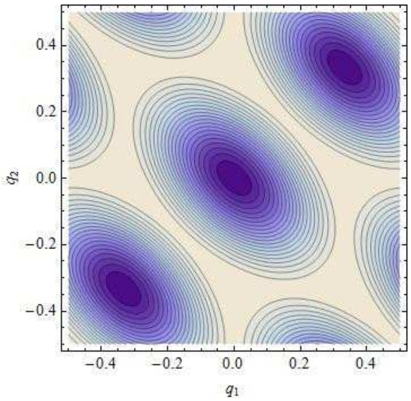

Fig. 5: The same figure as Fig. 3 but for the case of ADJ-APB.

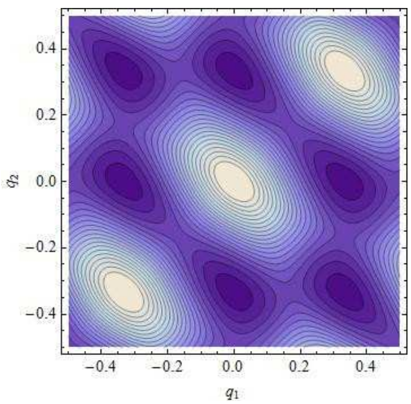

Fig. 6: The same figure as Fig. 3 but for the case of ADJ-PB.

PB fermions. The difference is important for the presence or absence of the GB in the total system, as seen later.

For the case of FTBC-PB, as shown in Fig. 8, $\mathcal{V}_f$ becomes minimum at $(q_1, q_2, q_3) = (\pm 1/9, \pm 1/9, \mp 2/9)$, $(\pm 2/9, \pm 2/9, \mp 4/3)$, $(\pm 4/9, \pm 4/3, \mp 8/9)$. The $\mathbb{Z}_3$ family of solutions yields a common value $|\Phi| = 0.577$, that is, the

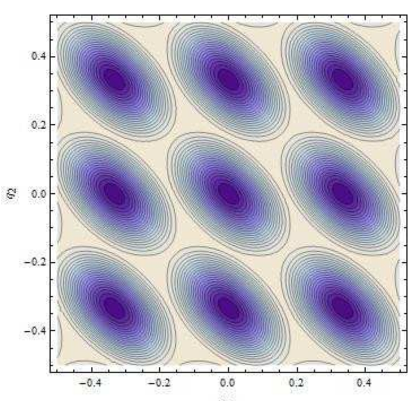

Fig. 7: The same figure as Fig. 3 but for the case of FTBC-APB.

GB takes place there. In the GB phase, $SU(3)$ gauge symmetry is broken to $SU(2) \times U(1)$ and $\mathbb{Z}_3$ and charge-conjugation symmetry are spontaneously broken.

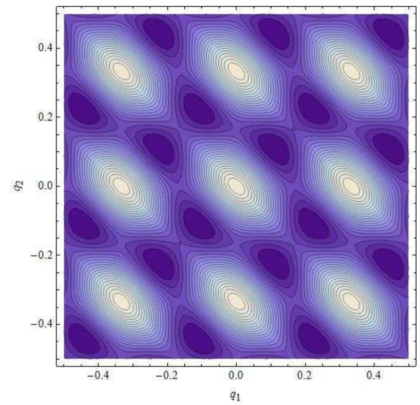

Fig. 8: The same figure as Fig. 3 but for the case of FTBC-PB.

Next we consider the gluon one-loop potential $\mathcal{V}_g$. As shown in Fig. 9, the potential $\mathcal{V}_g$ has minima at three solutions $(q_1, q_2, q_3) = (0,0,0), (\pm 1/3, \pm 1/3, \mp 2/3)$ leading to $|\Phi| = 1$. In the case of FTBC-APB, therefore, the GB does not take place when $\mathcal{V}_g$ is switched on, since $\mathcal{V}_g$ makes the solutions yielding $|\Phi| = 1$ more stable than the solutions yielding $\Phi = 0$. In the case of ADJ-PB, the re-confined phase appears even for $N_{F,adj} = 1$, since effects of $\mathcal{V}_f$ overcome those of $\mathcal{V}_g$ in the small mL limit, .

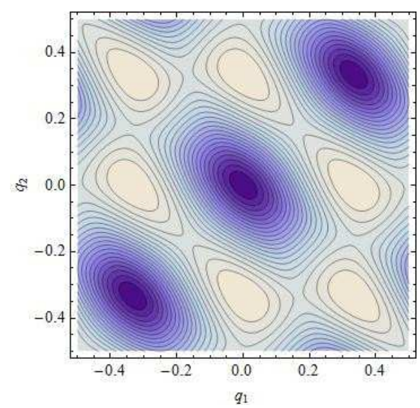

Fig. 9: Contour plot of $\mathcal{V}_g L^4$ in the q_1-q_2 plane. Here, q_3 is given by $-q_1 - q_2$.

In the case of FTBC-PB, the situation is complicated. If $N_{F,fund}$ is small, any nontrivial solution that induces the GB does not appear. In the case of large $N_{F,fund} \geq 30$, nontrivial solutions can survive at small mL, as shown in Fig. 10 (bottom). In the case of FTBC-PB, unlike the case of ADJ-PB, large $N_{F,fund}$ is required for nontrivial solutions to survive. This difference comes from the fact that $\mathcal{V}_f(\Phi = 1) - \mathcal{V}_f(\Phi = \Phi_{\min})$ is much smaller in the FTBC-PB case than that in the ADJ-PB case, where $\Phi_{\min}$ is a value of Φ at minimum points of $\mathcal{V}_f$.

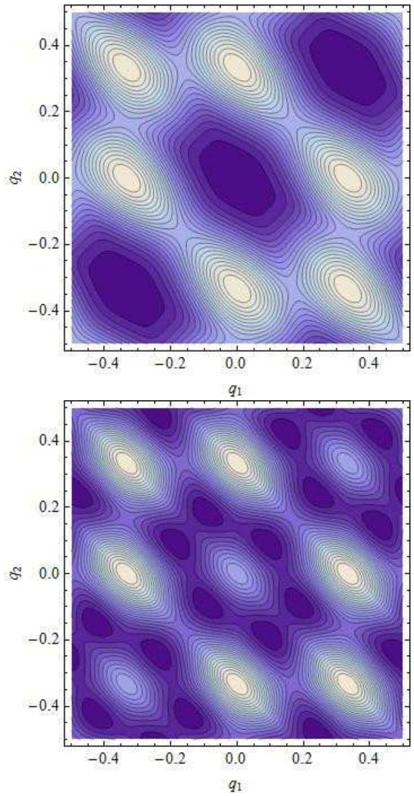

Fig. 10: Contour plot of $[\mathcal{V}_g+\mathcal{V}_f]L^4$ in the q_1-q_2 plane for the case of $N_{F,fund}=120$ FTBC fermions. The upper panel corresponds to the $SU(3)$ deconfined phase and the lower panel does to the $SU(2)\times U(1)$ C-broken phase.

For the case of FTBC-PB, the vacua corresponding to the broken phase are given by two $\mathbb{Z}_3$ families of solutions,

$$\begin{aligned}(q_1,q_2,q_3)_1 =&[(\alpha/9,\alpha/9,-2\alpha/9),\\ &((\alpha+3)/9,(\alpha+3)/9,(3-2\alpha)/9),\\ &(-(3-\alpha)/9,-(3-\alpha)/9,(6-2\alpha)/9)]\end{aligned} \tag{56}$$

$$\begin{aligned}(q_1,q_2,q_3)_2 =&[-(\alpha/9,\alpha/9,-2\alpha/9),\\ &-((\alpha+3)/9,(\alpha+3)/9,(3-2\alpha)/9),\\ &-(-(3-\alpha)/9,-(3-\alpha)/9,(6-2\alpha)/9)],\end{aligned} \tag{57}$$

where α as a function of $N_{F,fund}$ and mL varies in a range of $0<\alpha\le 1$. In the limit of $N_{F,fund}\to\infty$, α reaches 1. Each family of solutions has two other choices of permutation; for example, $(q_1,q_2,q_3)_{1'}=(\alpha/9,-2\alpha/9,\alpha/9)$ for the first solution of (56). In this GP phase, $SU(3)$ gauge symmetry is broken to $SU(2)\times U(1)$.

The two families of solutions, (56) and (57), are related to each other as

$$(q_1,q_2,q_3)_1=-(q_1,q_2,q_3)_2. \tag{58}$$

This comes from the fact that the system is invariant under charge conjugation $A_\mu\to -A_\mu$ and $\mathrm{Im}\Phi\to-\mathrm{Im}\Phi$ [35]. This charge conjugation symmetry is also spontaneously broken in this GB phase. Furthermore, $\mathbb{Z}_3$ symmetry is spontaneously violated in this phase, since $\Phi\neq 0$ there. A distribution plot of Φ is shown in Fig. 11, while the phase diagram is depicted in Fig. 12.

Here we comment on asymptotic non-freedom and renormalizability in a $SU(3)$ gauge theory with large $N_{F,fund}$. In the case with $N_{F,fund}\ge 30$, the theory may lose asymptotic freedom and is expected to become non-renormalizable. However, the dynamical GB due to the Hosotani mechanism can be brought about also in the asymptotic non-free theory as QED or five-dimensional gauge theories. This is because the mechanism is based on the Aharonov-Bohm effect in the compact direction. We thus may consider that our result on the GB may be still valid, although we should regard our large-flavor theory on $R^3\times S^1$ as a cutoff theory. There is no consensus on how the beta function behaves as a function of the number of flavors in a compactified gauge theory with special boundary conditions such as FTBC. Intensive study should be devoted to this topic.

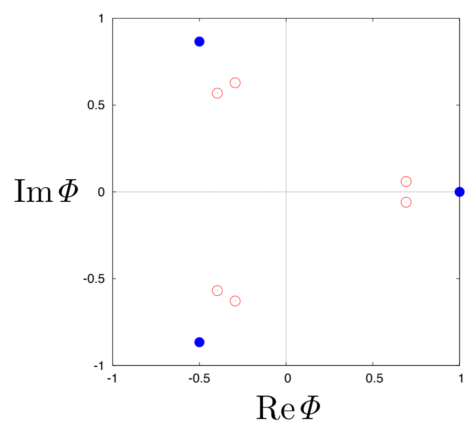

Fig. 11: Distribution of the Polyakov loop Φ in the complex plane for a $SU(3)$ gauge theory on $R^3\times S^1$ with $N_{F,fund}=120$ FTBC fermions. Solid circles correspond to the deconfinement phase and open circles do to the C-broken phase.

The situation is qualitatively similar also in five-dimensional gauge theories. In this case, the effective potentials (52)∼(55) are replaced by

$$\mathcal{V}_g=-\frac{9}{4\pi^2L^5}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{n=1}^{\infty}\Big(1-\frac{1}{3}\delta_{ij}\Big)\frac{\cos(2n\pi q_{ij})}{n^5}, \tag{59}$$

$$\begin{aligned}\mathcal{V}_{f,\mathrm{FD}} =& \frac{\sqrt{2}N_{F,fund}m^2}{\pi^{5/2}L^5}\sum_{i=1}^{3}\sum_{n=1}^{\infty}\frac{K_2(nmL)}{n^{5/2}}\\ &\times\cos[2\pi n(q_i+\varphi)],\end{aligned} \tag{60}$$

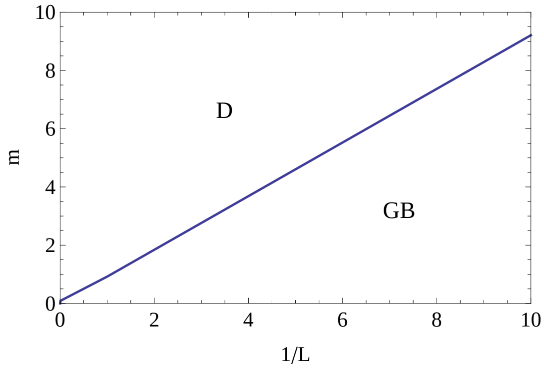

Fig. 12: The phase diagram in the L^{-1}-m plane for a $SU(3)$ gauge theory on $R^3 \times S^1$ with $N_{F,fund} = 120$ FTBC fermions. The symbol D stands for the $SU(3)$ deconfinement phase and GB for the $SU(2) \times U(1)$ gauge-symmetry broken phase. In the gauge-symmetry broken phase, charge conjugation is also spontaneously broken.

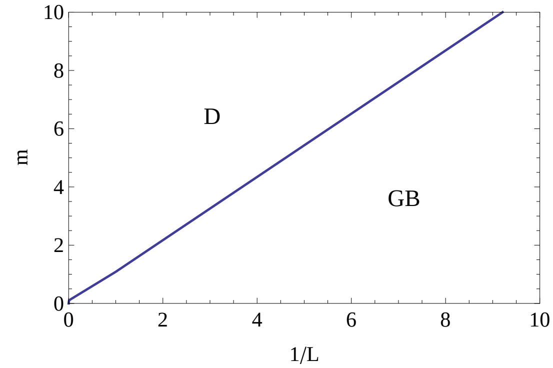

Fig. 13: The same figure as Fig. 12 but for a $SU(3)$ gauge theory on $R^4 \times S^1$ with $N_{F,fund} = 240$ FTBC fermions.

$$\mathcal{V}_{f,\mathrm{FTBC}} = \frac{\sqrt{2}N_3 m^2}{L^2\pi^{5/2}} \sum_{i=1}^{3}\sum_{f=1}^{3}\sum_{n=1}^{\infty} \times \frac{K_2(nmL)}{n^{5/2}} \cos[2\pi n(Q_{if}+\varphi)], \quad (61)$$

and

$$\mathcal{V}_{f,\mathrm{ADJ}} = \frac{\sqrt{2}N_{F,adj} m^2}{L^2\pi^{5/2}} \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{n=1}^{\infty}\Big(1-\frac{1}{3}\delta_{ij}\Big) \times \frac{K_2(nmL)}{n^{5/2}} \cos[2\pi n(q_{ij}+\varphi)], \quad (62)$$

respectively. Figure 13 shows the phase diagram for a five-dimensional gauge theory with FTBC-PB fundamental fermions in the case of $N_{F,fund} = 240$. Similarly to the four-dimensional case, the GB takes place for small L and large $N_{F,fund}$. However, the critical flavor number is $N_{F,fund} = 123$ much larger than that in the four-dimensional case. This difference is originated in two facts. The first is that the gauge degree of freedom is proportional to $d-2$ for d being dimensions of spacetime, while the fermion degree of freedom is to $2^{[d/2]}$, where the symbol "[ ]" denotes the Gauss symbol. Therefore, the gauge degree of freedom increases more rapidly than that of fermion when d increases. The second is more important. This is d dependence of the power of n in the denominators of $\mathcal{V}_f$ and $\mathcal{V}_g$. This dependence changes the range of n that mostly contributes the summation of n and makes $\mathcal{V}_g/\mathcal{V}_f$ larger as d increases.

V. SUMMARY

In summary, we have investigated differences and similarities between fundamental and adjoint matters in $SU(N)$ gauge theories. The gauge theory with ordinary fundamental matter does not have $\mathbb{Z}_N$ symmetry, whereas the gauge theory with adjoint (ADJ) matter does. This implies that an essential difference between fundamental and ADJ matters comes from the presence or absence of $\mathbb{Z}_N$ symmetry. We have then imposed the FTBC on fundamental fermion in order to make the gauge theory $\mathbb{Z}_N$ symmetric, and have shown similarities between FTBC fundamental matter and ADJ matter by using the PNJL model, particularly for the confinement/deconfinement transition related to $\mathbb{Z}_N$ symmetry. Thus a main difference between ordinary fundamental matter and ADJ matter is originated in the presence or absence of $\mathbb{Z}_N$ symmetry. Meanwhile, the chiral property is somewhat different between FTBC fundamental matter and ADJ matter, but has a simple scaling low $\sigma_{\mathrm{FTBC}} \sim \sigma_{\mathrm{ADJ}} N/N_{adj}$, where σ_{ADJ} (σ_{FTBC}) is the chiral condensate for ADJ (FTBC fundamental) fermion and N_{adj} (N) is the dimension of the adjoint (fundamental) representation of fermion and N_{adj} is related to N as $N_{adj} = N^2 - 1$.

We have also investigated a possibility of the gauge symmetry breaking (GB) at high-energy scale. The GB takes place for not only ADJ-PB fermion but also FTBC-PB fermion. Properties of the GB phase are different between the two fermions. At high-energy limit, color $SU(3)$ group is broken down to $U(1) \times U(1)$ for ADJ-PB fermion, but to $SU(2)\times U(1)$ for FTBC-PB fermion. For FTBC-PB fermion, the GB phase appears only when $N_{F,fund}$ is large and the charge-conjugation symmetry is also spontaneously broken there. The present results may suggest another class of Gauge-Higgs unification models due to the dynamical gauge symmetry breaking, although realization of the breaking at large $N_{F,fund}$ may mean that the system is not asymptotic free.

The BCFL transformation (9) plays an important role to make a gauge theory $\mathbb{Z}_N$-symmetric. $\mathbb{Z}_N$ symmetry, i.e., invariance under this linked transformation is originated in the fact that $\mathbb{Z}_N$ group is a common subgroup of $U(1)_{\mathrm{B}}$, color $SU(N)$ and flavor $SU(N)$. We can classify types of diquark condensates by using the BCFL transformation. This classification is an interesting future work.

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